Research Statement

The works of Parisi on spin glasses [Par79; Par80] and the *Volovich Hypothesis* [Vol10; Var11, Chapter 6] offer motivation for the study of non-Archimedean diffusion processes. Seminal works of Kochubei [Koc97] and Albeverio and Karwowski [AK94] introduced and developed the mathematical theory of non-Archimedean diffusion processes. Varadarajan discussed a generalization of classical Brownian motion valued in a very general family of non-Archimedean spaces [Var97]: Take K to be a local field of any characteristic[1], D a division ring that is finite-dimensional over K, and W a finite-dimensional D-module. Varadarajan's diffusion processes are W-valued. I take Varadarajan's work as my starting point, and study the processes, their components, their symmetries, their discrete approximations, and their applications. The non-Archimedean setting provides a wealth of interesting questions in Probability Theory and Stochastic Processes.

Every direct physical measurement results in a rational number. The real numbers are the traditional setting for analysis of these rational measurements. The general principle that motivates the study of Brownian motion in local field settings is that all completions of the rationals ought to be treated on even footing. Asking the same questions about both $\mathbb{R}$ and $\mathbb{Q}_p$, the p-adic numbers, for each prime p helps determine which properties of our processes are intrinsic, and which are artifacts of real number setting.

For a concrete example, there are two equivalent ways to define Brownian motion valued in $\mathbb{R}^d$. One approach (see, e.g. [SP12; KS98]) is to construct the finite-dimensional distributions from the fundamental solution to the d-dimensional heat equation

$$\partial_t u = \frac{\sigma}{2}\Delta u.$$

An alternative approach (see, eg. [Dur19; Law23]) is to define $W_t^{(i)}$ to be independent, identically-distributed $\mathbb{R}$-valued Brownian motions, and take the vector

$$\vec{W}_t = \left(W_t^{(1)}, \ldots, W_t^{(d)}\right)$$

to be the definition of $\mathbb{R}^d$-valued Brownian motion

These constructions are fail to be equivalent in the local field setting [RW23]:

Theorem 1 (Rajkumar, Weisbart) *The component processes of $\mathbb{Q}_p^d$-valued Brownian motion are stochastically dependent for all time.*

The equivalence of the constructions in the classical setting should be viewed as a consequence of $\mathbb{R}$. Despite this, the component processes are still themselves Brownian motions [RW23]:

Theorem 2 (Rajkumar, Weisbart) *The component processes of $\mathbb{Q}_p^d$-valued Brownian motion are identically-distributed $\mathbb{Q}_p$-valued Brownian motions.*

The component dependence reflects the underlying symmetries of $\mathbb{Q}_p^d$. The first exit times of the process from balls in $\mathbb{Q}_p^d$ ([RW23], Theorem 4.2) are heavily influenced by this dependence as compared to a process with independent components, which should be important in applications.

[1] K is a topological field which is isomorphic to either the real numbers, the complex numbers, a finite extension of the p-adic numbers for some prime p, or the field of Laurent polynomials over the field $\mathbb{F}_q$ for some prime power q (see [Mil20], Remark 7.49)

Our current work extends the prior work to the setting of finite-dimensional local fields over $\mathbb{Q}_p$ in order to better understand how Brownian motions captures the properties of the underlying spaces, in the spirit of Varadhan's formula [Var67]. The symmetries induced by the multiplicative structure are analogous to a choice of complex structure, and appear to nontrivially influence the component processes. The component processes are no longer necessarily identically distributed, though are still Brownian motions. To this end, we would like to achieve the following goal:

Goal: Determine the relationship between component processes of Brownian motion in Varadarajan's full generality.

A problem that Varadarajan suggested further motivates this goal. Quantum mechanics defined over general abelian groups goes back to Weyl [Wey09] and Schwinger [Sch70]. Varadarajan was motivated by studies of quantum systems over spaces analogous to the reals [DVV94; DHV99; BD15; BDW16], specifically the goal of understanding the spectra of and semigroups generated by operators of the form

$$H = \Delta_b + V.$$

Here, Δ_b is the analogue of the Laplacian (depending on a parameter $b > 0$) used to define the Brownian motion, and the potential V is a multiplication operator. Varadarajan suggests taking V to be the Coulomb potential, given by multiplication by $1/|x|$, and solving the corresponding Schrödinger equation. This is an analogue of the Coulomb problem, used to model the hydrogen atom [Tha05]. Although Varadarajan promised to consider the problem in a future paper, none was published prior to his passing.

Goal: Understand solutions to the local field Coulomb problem as posed by Varadarajan.

The most elementary setting in which Varadarajan suggests to study this problem is a particular module over a p-adic quaternion algebra. In order to accomplish this goal, we will need to understand the properties of Brownian motion in significantly greater generality. Our current project, that involves certain local fields which are algebras over $\mathbb{Q}_p$, is a step in this direction. However, these algebras are only two-dimensional, and moreover abelian. We will need to better understand how the more general, non-commutative algebra setting affects the Brownian motion before we can consider the Coulomb problem.

In addition to the component properties of p-adic Brownian motion, we study the scaling limit properties. It is well-known that classical Brownian motion is a scaling limit of a large class of random walks. However, it was only recently shown [BW19; Wei24] that similar results for p-adic Brownian motion. We extended the previous approximation framework to more general vector spaces V over local fields [Pie+24]. Quantities such as the mean and the variance determine classes of random walk that all converge to the same process under scaling limit in the classical setting. Such quantities are not available in the p-adic setting, and our results do not provide a framework for answering questions about universality. Instead, we witnessed Brownian motion in V as a scaling limit of a *particular* random walk.

Goal: Determine governing features of random walks that give rise to a Brownian motion in V under scaling limits.

Here we benefit from our ongoing project to understand the influence of multiplicative structure on Brownian motion, since we have multiple distinct Brownian motions living on the same underlying space. Our framework provides examples of random walks that converge under scaling to different

Brownian motions on the underlying $\mathbb{Q}_p$-vector spaces, depending on the multiplicative structure. We can compare these to determine which features are shared and which differ, allowing us to identify the properties that influence the limiting process.

An important fact about classical Brownian motion is Lévy's Forgery Theorem (see e.g. [Wen18]). The Forgery Theorem shows that the nowhere differentiable sample paths of Brownian motion approximate any given continuous curve with positive probability. The Onsager-Machlup functional of a stochastic process, introduced in [MO53; OM53], is a probabilistic analogue of the Lagrangian of a dynamical system [CW23]. The Onsager-Machlup functional provides a quantitative determination of how likely it is that the stochastic process will take a prescribed path. Applied to Brownian motion, this greatly strengthens the content of Lévy's Forgery Theorem and gives a quantitative measurement of how likely Brownian motion is to approximate a fixed curve. Determining the Onsager-Machlup functional for various stochastic processes is a well-known and widely studied problem in probability theory (see e.g. the discussion in [CG23]). A natural goal is then the following:

Goal: Determine the Onsager-Machlup functional for p-adic Brownian motion.

Our scaling limit framework should allow us to prove an analogue of Lévy's Forgery Theorem for p-adic Brownian motion. The basic framework of stochastic analysis in $\mathbb{Q}_p$ was defined in [Koc97], but to identify the Onsager-Machlup functional will likely require further utilizing and adapting tools of classical stochastic analysis to the p-adic setting. We believe that the choice of multiplicative structure imposed on the underlying space will also be evident in the resulting Onsager-Machlup functional.

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